The Bootstrap

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Math 243: Stat Learning

September 27th, 2021

Outline

In today's class, we will. . .

- Discuss the bootstrap for estimating variance of error
- Implement bootstrapping in R

Section 1

The Bootstrap

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  - Hope that the sample size is large enough to allow the Central Limit Theorem to come into play so that the statistic is approximately Normal

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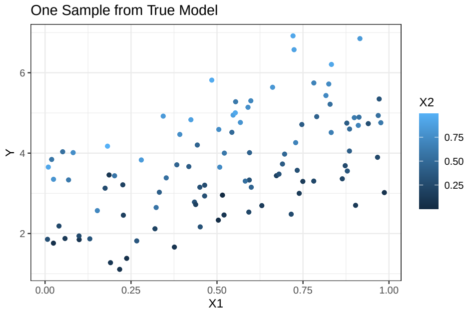
  - Assume that your sample is large enough to be “representative” of your population.
  - Create a new bootstrap sample by sampling **with replacement** from your original sample, a number of times equal to your original sample size.
  - Repeat the process to create many bootstrap samples. Compute the statistic of interest on each and plot the results.

Bootstrap Demo

Suppose $Y = 1 + 2 \cdot X_1 + 3 \cdot X_2 + X_1 \cdot X_2 + \epsilon$ with $\epsilon \sim N(0, 0.25)$.

```
set.seed(10101)
n<-100
X1<-runif(n, 0, 1)
X2 <- runif(n, 0, 1)
e<-rnorm(n, 0 ,.5)
Y<-1 + 2*X1 + 3*X2 + X1*X2+ e
d<-data.frame(X1, X2, Y)
```



Bootstrap Demo

```
my_mod<-lm(Y ~ X1*X2, data = d)
summary(my_mod)$coefficients
```

```
##              Estimate Std. Error  t value     Pr(>|t|)
## (Intercept) 1.447174  0.2100171 6.890742 5.807042e-10
## X1          1.317290  0.3803365 3.463485 7.982768e-04
## X2          2.405724  0.4102938 5.863417 6.404175e-08
## X1:X2       2.044325  0.7415455 2.756844 6.985948e-03
```

```
b3 <-  my_mod$coefficients[4]
```

The Simulation Approach

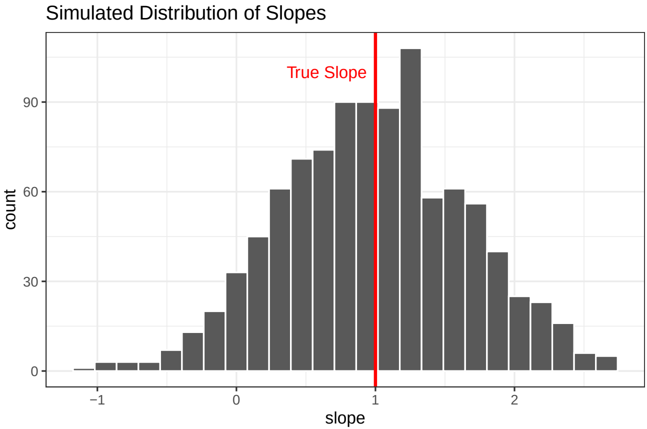
```
set.seed(234)
trials<-1000 #Number of simulations
n<-100 #Number points in each simulation
X1<-runif(n, 0, 1) # Generate random X1; same for all sims
X2 <- runif(n, 0, 1) # Generate random X1; same for all sims
slopes<-data.frame() #Create empty dataframe for the slopes

for (i in 1:trials){
sim_e<-rnorm(n, 0 ,.5)
sim_Y<-1 + 2*X1 + 3*X2 + X1*X2+ sim_e
sim_d<-data.frame(X1, X2, sim_Y)
sim_mod<-lm(sim_Y ~ X1*X2, data = sim_d)
slopes<-rbind( slopes,
               data.frame(slope = summary(sim_mod)$coefficients[4,1]))
}
```

```
head(slopes)
```

```
##        slope
## 1  0.5494089
## 2  1.4382129
## 3  0.9934332
## 4  0.7086642
## 5 -0.9140541
## 6  1.8136110
```

Simulation Distribution




```r
slopes %>% summarize(mean_slope = mean(slope), sd_slope = sd(slope))
```

```
##   mean_slope  sd_slope
## 1  0.9895467 0.6620953
```

The Bootstrap Approach

We have 1 sample:

```
head(d)
```

```
##          X1        X2        Y
## 1 0.1903066 0.1056760 1.275277
## 2 0.9108393 0.6749109 4.690218
## 3 0.2277161 0.1748862 2.455955
## 4 0.8249905 0.7360649 5.719890
## 5 0.9155760 0.8434911 6.849461
## 6 0.5052083 0.7491072 4.589090
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But we can create a bootstrap sample:

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set.seed(135)
a_bootstrap_sample<-slice_sample(d, n = n, replace = T)
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Duplicates?

```
common<-intersect(a_bootstrap_sample, d)
length(common$X1)
```

```
## [1] 66
```

The Bootstrap Approach, cont'd

Now, we create 1000 bootstraps and calculate the slope of each

```
# Create a function to compute statistic from bootstrap sample
set.seed(929)
interaction_slope <- function(split){
  x <- analysis(split)
  boot_mod <-lm(Y ~ X1*X2 , data = x)
  slope <- boot_mod$coefficients[4]
  slope
}
```

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}
```
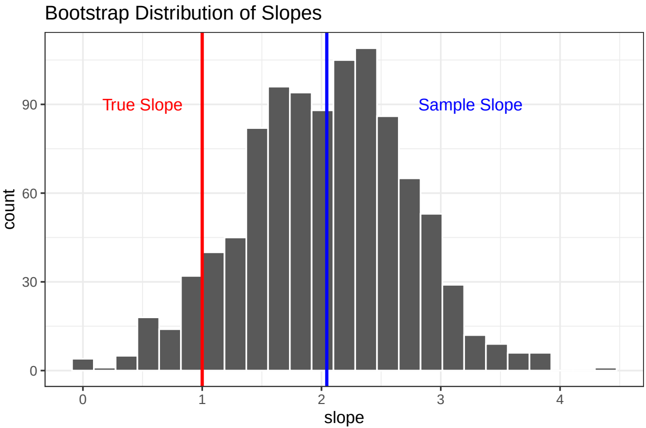

```
# Use rsample to create bootstrap samples and apply function
library(rsample)
bt_resamples <- bootstraps(d, times = 1000)
bt_resamples$slope <- map_dbl(bt_resamples$splits, interaction_slope)
```

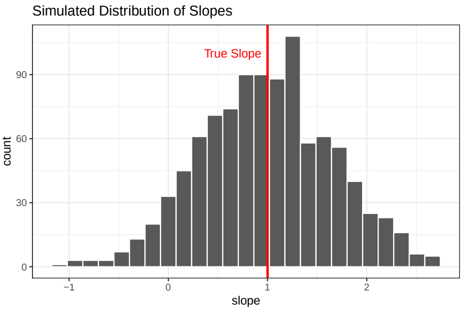
Bootstrap Distribution

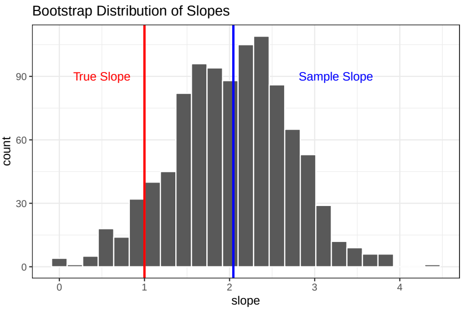



```
bt_resamples %>% summarize(mean_slope = mean(slope), sd_slope = sd(slope))
```

```
##   mean_slope  sd_slope
## 1   2.026826 0.6849343
```

Side-by-Side Comparison





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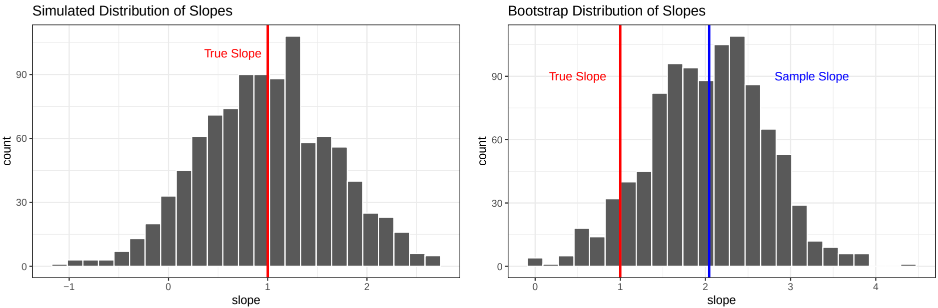


```
rbind(slopes, bt_resamples %>% select(slope)) %>%
  cbind(method = rep(c("sim", "boot"), each = 1000)) %>%
  group_by(method) %>% summarize(mean_slope = mean(slope), sd_slope = sd(slope),
                                 q.025 = quantile(slope,.025), q.975 = quantile(slope, .975))
```

```
## # A tibble: 2 x 5
##   method mean_slope sd_slope  q.025 q.975
##   <fct>       <dbl>    <dbl>  <dbl> <dbl>
## 1 boot        2.03     0.685  0.620  3.34
## 2 sim         0.990    0.662 -0.286  2.27
```

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Bootstrapping: Often used for *quantifying uncertainty*.

- Draw a bootstrap sample of size n from your data *with replacement*.
- Compute estimate of interest
- Consider distribution of bootstrap estimates over many samples